

Lower Decay Estimates for Non-Degenerate Kirchhoff Type Dissipative Wave Equations

By

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(Received September 30, 2018)*

Abstract

We consider the Cauchy problem for non-degenerate Kirchhoff type dissipative wave equations $\rho u'' + a(\|A^{1/2}u(t)\|^2)Au + u' = 0$ and $(u(0), u'(0)) = (u_0, u_1)$, where $u_0 \neq 0$. We derive the lower decay estimate $\|u(t)\|^2 \geq Ce^{-\beta t}$ for $t \geq 0$ with $\beta > 0$ for the solution $u(t)$.

2010 Mathematics Subject Classification. 35B40, 35L15

1 Introduction

Let H be a real Hilbert space with inner product (\cdot, \cdot) and norm $\|\cdot\|$. Let A be a linear operator on H with dense domain $\mathcal{D}(A)$. We assume that the operator A is self-adjoint and nonnegative such that $(Av, v) \geq 0$ for $v \in \mathcal{D}(A)$. The α -th power of A with dense domain $\mathcal{D}(A^\alpha)$ is denoted by A^α for $\alpha > 0$, and the graph-norm of A^α is denoted by $\|v\|_\alpha = (\|v\|^2 + \|A^\alpha v\|^2)^{\frac{1}{2}}$ for $v \in \mathcal{D}(A^\alpha)$. We use that $\|A^{1/2}v\|^2 = (Av, v)$ for $v \in \mathcal{D}(A^{1/2})$.

We study on the Cauchy problem for the non-degenerate Kirchhoff type dissipative wave equations :

$$\begin{cases} \rho u'' + a\left(\|A^{1/2}u(t)\|^2\right)Au + u' = 0, & t \geq 0 \\ (u(0), u'(0)) = (u_0, u_1) \in \mathcal{D}(A) \times \mathcal{D}(A^{1/2}), \end{cases} \quad (1.1)$$

where $u = u(t)$ is an unknown real value function, $' = d/dt$, and ρ is a positive constant.

For the non-local nonlinear term $a(M) \in C^0([0, \infty)) \cap C^1((0, \infty))$, we assume that as follows :

$$\text{Hyp.1} \quad K_1 \leq a(M) \leq K_2 + K_3 M^\gamma \quad \text{for } M \geq 0$$

$$\text{Hyp.2} \quad 0 \leq a'(M)M \leq K_4 a(M) \quad \text{for } M > 0$$

with $\gamma > 0$ and $K_j > 0$ ($j = 1, 2, 3, 4$).

From Hyp.1, we see that

$$K_1 M \leq \int_0^M a(\mu) d\mu \leq \left(K_2 + \frac{K_3}{\gamma + 1} M^\gamma \right) M. \quad (1.2)$$

For typical examples, we have that

$$a(M) = 1 + M^\gamma, \quad (1 + M)^\gamma, \quad \log(2 + M^\gamma).$$

In the case of one dimension, (1.1) describes small amplitude vibrations of an elastic string (see [3], [4], [6]).

We obtain the following global existence theorem (see Theorem 4.1 and Proposition 5.1).

Theorem 1.1 *Suppose that Hyp.1 and Hyp.2 are fulfilled. If the initial data (u_0, u_1) belong to $\mathcal{D}(A) \times \mathcal{D}(A^{1/2})$ and satisfy $u_0 \neq 0$, and moreover, the coefficient ρ and the initial data (u_0, u_1) satisfy the smallness condition (4.1), then the problem (1.1) admits a unique global solution $u(t)$ in the class*

$$C^0([0, \infty); \mathcal{D}(A)) \cap C^1([0, \infty); \mathcal{D}(A^{1/2})) \cap C^2([0, \infty); H).$$

Moreover, the solution $u(t)$ satisfies

$$\|A^{1/2}u(t)\|^2 \geq Ce^{-\alpha t} \quad \text{for } t \geq 0 \quad (1.3)$$

with some $\alpha > 0$.

In previous paper [10], we have derived the upper decay estimates of the solution $u(t)$ of (1.1) in the case of $a(M) = (1 + M)^\gamma$ with $\gamma > 0$ and $A = -\Delta = -\sum_{j=1}^N \partial^2 / \partial x_j^2$ with domain $\mathcal{D}(A) = H^2(\mathbb{R}^N)$:

$$\begin{aligned} \|A^{1/2}u(t)\|^2 &\leq C(1+t)^{-1}, \quad \|u'(t)\|^2 + \|Au(t)\|^2 \leq C(1+t)^{-2}, \\ \|A^{1/2}u'(t)\|^2 + \|u''(t)\|^2 &\leq C(1+t)^{-3} \quad \text{for } t \geq 0 \end{aligned}$$

(see [5], [8] for $a(M) = 1 + M^\gamma$ with $\gamma \geq 1$, that is, $a(\cdot) \in C^1([0, \infty))$).

On the other hand, Ghisi and Gobbino [5] have derived the lower decay estimate (1.3) for (1.1) (see [9] for bounded domains).

The purpose of this paper is to derive the lower decay estimate for $\|u(t)\|^2$.

For the non-local nonlinear term $a(M) \in C([0, \infty)) \cap C^2((0, \infty))$, we assume that as follows :

Hyp.3 $|a''(M)|M^2 \leq K_5 a(M)$ for $M > 0$

with $K_5 > 0$.

We obtain the following lower decay estimate of the solution $u(t)$ of (1.1) (see Theorem 5.4). Our main result is as follows.

Theorem 1.2 *Suppose that the assumption of Theorem 1.1 and Hyp.3 are fulfilled. Then, the solution $u(t)$ satisfies*

$$\|u(t)\|^2 \geq C e^{-\beta t} \quad \text{for } t \geq 0 \quad (1.4)$$

with some $\beta > 0$.

The notations we use in this paper are standard. Positive constants will be denoted by C and will change from line to line.

2 Local Existence and Energy

We have the following local existence theorem by standard arguments (see [1], [2], [7], [11] and the references cited therein).

Proposition 2.1 *Suppose that Hyp.1 and Hyp.2 are satisfied. If the initial data (u_0, u_1) belong to $\mathcal{D}(A) \times \mathcal{D}(A^{1/2})$, then the problem (1.1) admits a unique local solution $u(t)$ in the class $C^0([0, T]; \mathcal{D}(A)) \cap C^1([0, T]; \mathcal{D}(A^{1/2})) \cap C^2([0, T]; H)$ for some $T = T(\|u_0\|_2, \|u_1\|_1) > 0$.*

Moreover, $\|u(t)\|_2 + \|u'(t)\|_1 < \infty$ for $t \geq 0$, then we can take $T = \infty$.

In what follows, let $u(t)$ be a solution of (1.1) under the assumption of Proposition 2.1.

We set that

$$M(t) = \|A^{1/2}u(t)\|^2 \quad (2.1)$$

and

$$E(t) = \rho \|u'(t)\|^2 + \int_0^{M(t)} a(\mu) d\mu \quad (2.2)$$

for simplicity of the notations.

Proposition 2.2 *Under the assumption of Proposition 2.1, the solution $u(t)$ of (1.1) satisfies that*

$$E(t) + 2 \int_0^t \|u'(s)\|^2 ds = E(0), \quad (2.3)$$

$$M(t) \leq K_1^{-1} E(0), \quad (2.4)$$

$$a(M(t)) \leq K_2 + K_3 (K_1^{-1} E(0))^\gamma \quad (\equiv I(0)), \quad (2.5)$$

$$\|u(t)\|^2 \leq 6(\|u_0\|^2 + \rho E(0)). \quad (2.6)$$

for $t \geq 0$.

Proof. Taking the inner product of (1.1) with $2u'(t)$, we have

$$\frac{d}{dt} E(t) + 2\|u'(t)\|^2 = 0, \quad (2.7)$$

and integrating (2.7) in time t , we obtain (2.3).

Moreover, it follows from (5.1) and (2.2) that

$$K_1 M(t) \leq E(t) \leq E(0),$$

and from Hyp.2 that

$$a(M(t)) \leq K_2 + K_3 M(t)^\gamma \leq I(0).$$

Taking the inner product of (1.1) with $u(t)$, we have

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|^2 + a(M(t))M(t) = \rho \left(\|u'(t)\|^2 - \frac{d}{dt} (u'(t), u(t)) \right),$$

and we observe from the Young inequality that

$$\begin{aligned} & \|u(t)\|^2 + 2 \int_0^t a(M(s))M(s) ds \\ & \leq \|u_0\|^2 + 2\rho \int_0^t \|u'(s)\|^2 ds + \|u_0\|^2 + \rho \|u_1\|^2 + \frac{1}{2} \|u(t)\|^2 + 2\rho^2 \|u'(t)\|^2 \end{aligned}$$

and hence

$$\begin{aligned} & \frac{1}{2} \|u(t)\|^2 + 2 \int_0^t a(M(s))M(s) ds \\ & \leq 2\|u_0\|^2 + \rho \left(\rho \|u_1\|^2 + 2\rho \|u'(t)\|^2 + 2 \int_0^t \|u'(s)\|^2 ds \right) \\ & \leq 2\|u_0\|^2 + 3\rho E(0) \end{aligned}$$

which implies the desired estimate (2.6). \square

3 Several Estimates

In order to obtain a-priori estimates of the solution $u(t)$, we assume that

$$\rho \frac{|M'(t)|}{M(t)} \leq \frac{1}{K_4 + 1} \quad (3.1)$$

where $M(t)$ is defined by (5.1).

Proposition 3.1 *Under the assumption (3.1), the solution $u(t)$ satisfies*

$$\frac{\|Au(t)\|^2}{M(t)} \leq G(t) \leq G(0), \quad (3.2)$$

where

$$G(t) = \frac{\|Au(t)\|^2}{M(t)} + \rho Q(t), \quad (3.3)$$

$$Q(t) = \frac{\|A^{1/2}u'(t)\|^2 \|A^{1/2}u(t)\|^2 - |(A^{1/2}u'(t), A^{1/2}u(t))|^2}{a(M(t))M(t)^2} \quad (\geq 0). \quad (3.4)$$

Proof. We have from (1.1) that

$$\begin{aligned} & \frac{d}{dt} \frac{\|Au(t)\|^2}{M(t)} \\ &= \frac{1}{a(M(t))M(t)^2} (2(a(M(t))Au, Au')M(t) - (a(M(t))Au, Au)M'(t)) \\ &= \frac{1}{a(M(t))M(t)^2} \left(2(\|A^{1/2}u'\|^2 + \rho(A^{1/2}u'', A^{1/2}u'))M(t) \right. \\ & \quad \left. - \left(\frac{1}{2}|M'(t)|^2 + \rho \left(\|A^{1/2}u'(t)\|^2 - \frac{1}{2}M''(t) \right) M'(t) \right) \right) \\ &= -2Q(t) + \rho R(t) \end{aligned}$$

where we set

$$R(t) = \frac{2(A^{1/2}u'', A^{1/2}u')M(t) + (\|A^{1/2}u'(t)\|^2 - \frac{1}{2}M''(t)) M'(t)}{a(M(t))M(t)^2}.$$

Since we observe

$$\begin{aligned}
& \frac{d}{dt}Q(t) \\
&= -\frac{a'(M(t))M'(t)M(t)^2 + 2a(M(t))M(t)M'(t)}{(a(M(t))M(t)^2)^2} \\
&\quad \times \left(\|A^{1/2}u'\|^2 M(t) - \frac{1}{4}|M'(t)|^2 \right) \\
&\quad + \frac{2(A^{1/2}u'', A^{1/2}u')M(t) + \|A^{1/2}u'\|^2 M'(t) - \frac{1}{2}M'(t)M''(t)}{a(M(t))M(t)^2} \\
&= -\frac{M'(t)}{M(t)} \frac{a'(M(t))M(t) + 2a(M(t))}{a(M(t))^2 M(t)^2} \left(\|A^{1/2}u'\|^2 M(t) - \frac{1}{4}|M'(t)|^2 \right) + R(t) \\
&= -\frac{M'(t)}{M(t)} \left(2 + \frac{a'(M(t))M(t)}{a(M(t))} \right) Q(t) + R(t),
\end{aligned}$$

we have

$$\begin{aligned}
& \frac{d}{dt} \left(\frac{\|Au(t)\|^2}{M(t)} + \rho Q(t) \right) \\
& \quad + 2 \left(1 + \frac{\rho M'(t)}{2 M(t)} \left(2 + \frac{a'(M(t))M(t)}{a(M(t))} \right) \right) Q(t) = 0.
\end{aligned}$$

Moreover, we observe

$$1 + \frac{\rho M'(t)}{2 M(t)} \left(2 + \frac{a'(M(t))M(t)}{a(M(t))} \right) \geq 1 - \frac{1}{2} \frac{1}{K_4 + 1} (2 + K_4) \geq 0$$

and $Q(t) \geq 0$, we have

$$\frac{d}{dt}G(t) = \frac{d}{dt} \left(\frac{\|Au(t)\|^2}{M(t)} + \rho Q(t) \right) \leq 0$$

which implies the desired estimate (3.2). \square

Proposition 3.2 *Under the assumption (3.1), the solution $u(t)$ satisfies*

$$\frac{\|u'(t)\|^2}{M(t)} \leq B(0), \tag{3.5}$$

where

$$B(0) = \max \left\{ \frac{\|u_1\|^2}{M(0)}, \frac{K_4 + 1}{K_4} I(0)^2 G(0) \right\}. \tag{3.6}$$

Proof. Taking the inner product of (1.1) with $2u'(t)/M(t)$, we have

$$\begin{aligned} \rho \frac{d}{dt} \frac{\|u'(t)\|^2}{M(t)} + \left(2 + \rho \frac{M'(t)}{M(t)}\right) \frac{\|u'(t)\|^2}{M(t)} &= -a(M(t)) \frac{M'(t)}{M(t)} \\ &\leq 2a(M(t)) \frac{\|Au(t)\| \|u'(t)\|}{M(t)} \\ &\leq a(M(t))^2 \frac{\|Au(t)\|^2}{M(t)} + \frac{\|u'(t)\|^2}{M(t)} \end{aligned} \quad (3.7)$$

where we used the Young inequality.

Since

$$1 + \rho \frac{M'(t)}{M(t)} \geq \frac{K_4}{K_4 + 1} \quad \text{and} \quad a(M(t))^2 \frac{\|Au(t)\|^2}{M(t)} \leq I(0)^2 G(0),$$

we have

$$\rho \frac{d}{dt} \frac{\|u'(t)\|^2}{M(t)} + \frac{K_4}{K_4 + 1} \frac{\|u'(t)\|^2}{M(t)} \leq I(0)^2 G(0)$$

and hence, we obtain (3.6). \square

Remark. If the nonnegative function $f(t)$ satisfies

$$f'(t) + af(t) \leq b, \quad t \geq 0$$

with positive constants a and b , then

$$f(t) \leq \max\{f(0), b/a\}, \quad t \geq 0.$$

Indeed, taking

$$g(t) = \max\{f(0), b/a\}, \quad t \geq 0,$$

we see that $-ag(t) + b \leq 0$ and $g'(t) = 0$, and hence,

$$g'(t) + ag(t) \geq b \quad \text{and} \quad f(0) \leq g(0).$$

Thus, by the comparison principle, we conclude.

4 Global Existence

Theorem 4.1 *Suppose that Hyp.1 and Hyp.2 are fulfilled. If the initial data (u_0, u_1) belong to $\mathcal{D}(A) \times \mathcal{D}(A^{1/2})$ are satisfies $u_0 \neq 0$ and*

$$2\rho G(0)^{\frac{1}{2}} B(0)^{\frac{1}{2}} < \frac{1}{K_4 + 1}, \quad (4.1)$$

then the problem (1.1) admits a unique global solution $u(t)$ in the class

$$C^0([0, \infty); \mathcal{D}(A)) \cap C^1([0, \infty); \mathcal{D}(A^{1/2})) \cap C^2([0, \infty); H)$$

and the solution $u(t)$ satisfies

$$\|u(t)\|^2 \leq 6(\|u_0\|^2 + \rho E(0)), \quad (4.2)$$

$$E(t) \leq E(0), \quad a(M(t)) \leq I(0), \quad (4.3)$$

$$\rho \frac{|M'(t)|}{M(t)} \leq \frac{1}{K_4 + 1}, \quad (4.4)$$

$$\frac{\|Au(t)\|^2}{M(t)} \leq G(0), \quad \frac{\|u'(t)\|^2}{M(t)} \leq B(0). \quad (4.5)$$

Proof. Let $u(t)$ be a solution on $[0, T]$. Since we observe from (3.2), (3.5), and (4.1) that

$$\rho \frac{|M'(0)|}{M(0)} \leq 2\rho \frac{\|u_1\|}{M(0)^{\frac{1}{2}}} \frac{\|Au_0\|}{M(0)^{\frac{1}{2}}} \leq 2\rho B(0)^{\frac{1}{2}} G(0) < \frac{1}{K_4 + 1},$$

putting

$$T = \sup\{t \in [0, \infty) \mid \rho \frac{|M'(s)|}{M(s)} < \frac{1}{K_4 + 1} \text{ for } 0 \leq s < t\},$$

we see that $T_1 > 0$. If $T_1 < T$, we have

$$\rho \frac{|M'(t)|}{M(t)} < \frac{1}{K_4 + 1} \quad \text{for } 0 \leq t < T_1, \quad \text{and} \quad \rho \frac{|M'(T_1)|}{M(T_1)} = \frac{1}{K_4 + 1}.$$

Again, from (3.2), (3.5), and (4.1) it follows that

$$\rho \frac{|M'(t)|}{M(t)} \leq 2\rho \frac{\|u'(t)\|}{M(t)^{\frac{1}{2}}} \frac{\|Au(t)\|}{M(t)^{\frac{1}{2}}} \leq 2\rho B(0)^{\frac{1}{2}} G(0) < \frac{1}{K_4 + 1}$$

for $0 \leq t \leq T$, and hence, we obtain $T_1 \geq T$, and we see that the solution $u(t)$ satisfies the estimates (2.3)–(2.6), (3.2), and (3.5), which implies (4.2)–(4.5).

Taking the inner product of (1.1) with $2Au'(t)/a(M(t))$, we have

$$\begin{aligned} & \frac{d}{dt} \left(\rho \frac{\|A^{1/2}u'(t)\|^2}{a(M(t))} + \|Au(t)\|^2 \right) \\ & + 2 \left(1 + \frac{\rho a'(M(t))M(t)}{a(M(t))} \frac{M'(t)}{M(t)} \right) \frac{\|A^{1/2}u'(t)\|^2}{a(M(t))} = 0. \end{aligned}$$

Since

$$\begin{aligned} 1 + \frac{\rho a'(M(t))M(t)}{a(M(t))} \frac{M'(t)}{M(t)} & \geq 1 - \frac{K_4}{2} \rho \frac{|M'(t)|}{M(t)} \\ & \geq 1 - \frac{K_4}{2} \frac{1}{K_4 + 1} \geq 0, \end{aligned}$$

we have

$$\frac{d}{dt} \left(\rho \frac{\|A^{1/2}u'(t)\|^2}{a(M(t))} + \|Au(t)\|^2 \right) \leq 0,$$

and hence,

$$\|A^{1/2}u'(t)\|^2 + \|Au(t)\|^2 \leq C \quad \text{for } 0 \leq t \leq T.$$

Thus, we observe that $\|u(t)\|_2 + \|u'(t)\|_1 \leq C$, and by the second statement of Proposition 2.1, we conclude that the problem (1.1) admits a unique global solution. \square

5 Lower Decay Estimates

Proposition 5.1 *Under the assumption of Theorem 4.1, it holds that*

$$M(t) \geq Ce^{-\alpha t} \quad \text{for } t \geq 0 \quad (5.1)$$

with some $\alpha > 0$.

Proof. Taking the inner product of (1.1) with $2u'(t)/M(t)^2$, we have

$$\begin{aligned} & \frac{d}{dt} \left(\rho \frac{\|u'(t)\|^2}{M(t)^2} + \frac{a(M(t))}{M(t)} \right) + 2 \left(1 + \rho \frac{M'(t)}{M(t)} \right) \frac{\|u'(t)\|^2}{M(t)^2} \\ &= \frac{-2a(M(t)) + a'(M(t))M(t)}{M(t)} \frac{M'(t)}{M(t)} \\ &\leq C \frac{a(M(t))}{M(t)} \frac{M'(t)}{M(t)} \leq \alpha \frac{a(M(t))}{M(t)} \end{aligned}$$

with some $\alpha > 0$, where we used Hyp.2 and (4.4).

Since $1 + \rho M'(t)/M(t) \geq 0$, we have

$$\frac{d}{dt} \left(\rho \frac{\|u'(t)\|^2}{M(t)^2} + \frac{a(M(t))}{M(t)} \right) \leq \alpha \left(\rho \frac{\|u'(t)\|^2}{M(t)^2} + \frac{a(M(t))}{M(t)} \right)$$

and hence, we obtain

$$\rho \frac{\|u'(t)\|^2}{M(t)^2} + \frac{a(M(t))}{M(t)} \leq Ce^{\alpha t} \quad \text{or} \quad M(t) \geq Ce^{-\alpha t}$$

where we used the assumption that $a(M(t)) \geq K_1 > 0$. \square

Proposition 5.2 *Under the assumption of Theorem 4.1, it holds that*

$$\frac{\|A^{1/2}u'(t)\|^2}{M(t)} \leq C \quad \text{for } t \geq 0. \quad (5.2)$$

Proof. Taking the inner product of (1.1) with $(2Au'(t) + \rho^{-1}Au(t))/M(t)$, we have

$$\begin{aligned} & \frac{d}{dt} \left(\rho \frac{\|A^{1/2}u'(t)\|^2}{M(t)} + a(M(t)) \frac{\|Au(t)\|^2}{M(t)} + \frac{(Au(t), u'(t))}{M(t)} \right) \\ & + \left(1 + \rho \frac{M'(t)}{M(t)} \right) \frac{\|A^{1/2}u'(t)\|^2}{M(t)} + \frac{a(M(t))}{\rho} \frac{\|Au(t)\|^2}{M(t)} + \frac{1}{2} \frac{|M'(t)|^2}{M(t)^2} \\ & = - (a(M(t)) + a'(M(t))M(t)) \frac{M'(t)}{M(t)} \frac{\|Au(t)\|^2}{M(t)} - \frac{1}{2\rho} \frac{M'(t)}{M(t)}. \end{aligned} \quad (5.3)$$

Moreover, taking (5.3) + (3.7) $\times \rho^{-1}K_1^{-1}$, we have

$$\begin{aligned} & \frac{d}{dt} F(t) + \left(1 + \rho \frac{M'(t)}{M(t)} \right) \frac{\|A^{1/2}u'(t)\|^2}{M(t)} + \frac{a(M(t))}{\rho} \frac{\|Au(t)\|^2}{M(t)} + \frac{1}{2} \frac{|M'(t)|^2}{M(t)^2} \\ & + \frac{1}{\rho K_1} \left(2 + \rho \frac{M'(t)}{M(t)} \right) \frac{\|u'(t)\|^2}{M(t)} = R(t) \end{aligned}$$

where

$$\begin{aligned} F(t) &= H(t) + \frac{(Au(t), u'(t))}{M(t)}, \\ H(t) &= \rho \frac{\|A^{1/2}u'(t)\|^2}{M(t)} + a(M(t)) \frac{\|Au(t)\|^2}{M(t)} + \frac{1}{K_1} \frac{\|u'(t)\|^2}{M(t)} \quad (\geq 0), \\ R(t) &= - (a(M(t)) + a'(M(t))M(t)) \frac{M'(t)}{M(t)} \frac{\|Au(t)\|^2}{M(t)} - \frac{1}{2\rho} \frac{M'(t)}{M(t)} \\ & \quad - \frac{a(M(t))}{\rho K_1} \frac{M'(t)}{M(t)}. \end{aligned}$$

Since we observe from the Young inequality and Hyp.1 that

$$\begin{aligned} \frac{|(Au(t), u'(t))|}{M(t)} &\leq \frac{K_1}{2} \frac{\|Au(t)\|^2}{M(t)} + \frac{1}{2K_1} \frac{\|u'(t)\|^2}{M(t)} \\ &\leq \frac{a(M(t))}{2} \frac{\|Au(t)\|^2}{M(t)} + \frac{1}{2K_1} \frac{\|u'(t)\|^2}{M(t)}, \end{aligned}$$

and from (4.4) that

$$1 + \rho \frac{M'(t)}{M(t)} \geq \frac{K_4}{K_4 + 1} \quad (> 0)$$

and from (4.3)–(4.5) that

$$|R(t)| \leq C \frac{|M'(t)|}{M(t)} \leq C,$$

we have

$$\frac{d}{dt}F(t) + \nu F(t) \leq C$$

with some $\nu > 0$, and hence,

$$F(t) \leq C \quad \text{or} \quad H(t) \leq C$$

which implies the desired estimate (5.2). \square

Proposition 5.3 *Under the assumption of Theorem 4.1 and Hyp.3, it holds that*

$$\frac{\|u''(t)\|^2}{M(t)} \leq C \quad \text{for } t \geq 0. \quad (5.4)$$

Proof. Taking the inner product of (1.1) with $(2u''(t) + \rho^{-1}u'(t))/M(t)$, we have

$$\begin{aligned} & \frac{d}{dt} \left(\rho \frac{\|u''(t)\|^2}{M(t)} + a(M(t)) \frac{\|A^{1/2}u'(t)\|^2}{M(t)} + \frac{a'(M(t))M(t)}{2} \frac{|M'(t)|^2}{M(t)^2} \right) \\ & + \frac{1}{2\rho} \frac{\|u'(t)\|^2}{M(t)} + \frac{(u''(t), u'(t))}{M(t)} \Bigg) + \left(1 + \rho \frac{M'(t)}{M(t)} \right) \frac{\|u''(t)\|^2}{M(t)} \\ & + \frac{a(M(t))}{\rho} \frac{\|A^{1/2}u'(t)\|^2}{M(t)} + \frac{a'(M(t))M(t)}{2\rho} \frac{|M'(t)|^2}{M(t)^2} \\ & = (-a(M(t)) + 3a'(M(t))M(t)) \frac{M'(t)}{M(t)} \frac{\|A^{1/2}u'(t)\|^2}{M(t)} \\ & + \frac{1}{2} (-a'(M(t))M(t) + a''(M(t))M(t)^2) \left(\frac{M'(t)}{M(t)} \right)^3 \\ & - \frac{M'(t)}{M(t)} \left(\frac{1}{2\rho} \frac{\|u'(t)\|^2}{M(t)} + \frac{(u''(t), u'(t))}{M(t)} \right). \end{aligned} \quad (5.5)$$

Moreover, taking (5.5)+(3.7), we have

$$\begin{aligned} & \frac{d}{dt}G(t) + \left(1 + \rho \frac{M'(t)}{M(t)} \right) \frac{\|u''(t)\|^2}{M(t)} + \frac{a(M(t))}{\rho} \frac{\|A^{1/2}u'(t)\|^2}{M(t)} \\ & + \frac{a'(M(t))M(t)}{2\rho} \frac{|M'(t)|^2}{M(t)^2} + \left(2 + \rho \frac{M'(t)}{M(t)} \right) \frac{\|u'(t)\|^2}{M(t)} = S(t) \end{aligned}$$

where

$$\begin{aligned}
G(t) &= K(t) + \frac{(u''(t), u'(t))}{M(t)}, \\
K(t) &= \rho \frac{\|u''(t)\|^2}{M(t)} + a(M(t)) \frac{\|A^{1/2}u'(t)\|^2}{M(t)} + \frac{a'(M(t))M(t)}{2} \frac{|M'(t)|^2}{M(t)^2} \\
&\quad + \left(\frac{1}{2\rho} + \rho \right) \frac{\|u'(t)\|^2}{M(t)}, \\
S(t) &= (-a(M(t)) + 3a'(M(t))M(t)) \frac{M'(t)}{M(t)} \frac{\|A^{1/2}u'(t)\|^2}{M(t)} \\
&\quad + \frac{1}{2} (-a'(M(t))M(t) + a''(M(t))M(t)^2) \left(\frac{M'(t)}{M(t)} \right)^3 \\
&\quad - \frac{M'(t)}{M(t)} \left(\frac{1}{2\rho} \frac{\|u'(t)\|^2}{M(t)} + \frac{(u''(t), u'(t))}{M(t)} \right) - a(M(t)) \frac{M'(t)}{M(t)}.
\end{aligned}$$

Since we observe from the Young inequality that

$$\frac{|(u''(t), u'(t))|}{M(t)} \leq \frac{\rho}{2} \frac{\|u''(t)\|^2}{M(t)} + \frac{1}{2\rho} \frac{\|u'(t)\|^2}{M(t)}$$

and from (4.4) that

$$1 + \rho \frac{M'(t)}{M(t)} \geq \frac{K_4}{K_4 + 1} \quad (> 0)$$

and from (4.3)–(4.5), (5.2) that

$$|S(t)| \leq C + \frac{K_4}{2(K_4 + 1)} \frac{\|u'(t)\|^2}{M(t)},$$

we have

$$\frac{d}{dt}G(t) + \nu G(t) \leq C$$

with some $\nu > 0$, and hence,

$$G(t) \leq 0 \quad \text{or} \quad K(t) \leq 0$$

which implies the desired estimate (5.4). \square

Theorem 5.4 *Suppose that the assumption of Theorem 4.1 and Hyp.3 are fulfilled. Then, the solution $u(t)$ satisfies*

$$\|u(t)\|^2 \geq Ce^{-\beta t} \quad \text{for } t \geq 0 \quad (5.6)$$

with some $\beta \geq \alpha > 0$.

Proof. Using (1.1), we observe that

$$\begin{aligned} \frac{d}{dt} \frac{M(t)}{\|u(t)\|^2} &= \frac{1}{\|u(t)\|^4} (2(Au(t), u'(t))\|u(t)\|^2 - 2M(t)(u(t), u'(t))) \\ &= \frac{-2}{\|u(t)\|^2} \left(\rho(Au(t) - \frac{M(t)}{\|u(t)\|^2}u(t), u''(t)) \right. \\ &\quad \left. + a(M(t))((Au(t) - \frac{M(t)}{\|u(t)\|^2}u(t), Au(t))) \right) \end{aligned}$$

and

$$(Au(t) - \frac{M(t)}{\|u(t)\|^2}u(t), Au(t)) = \|Au(t) - \frac{M(t)}{\|u(t)\|^2}u(t)\|^2.$$

Thus, we have

$$\begin{aligned} &\frac{d}{dt} \frac{M(t)}{\|u(t)\|^2} + \frac{2a(M(t))}{\|u(t)\|^2} \|Au(t) - \frac{M(t)}{\|u(t)\|^2}u(t)\|^2 \\ &= \frac{-2\rho}{\|u(t)\|^2} \rho(Au(t) - \frac{M(t)}{\|u(t)\|^2}u(t), u''(t)) \\ &\leq 2\rho \frac{1}{\|u(t)\|} \|Au(t) - \frac{M(t)}{\|u(t)\|^2}u(t)\| \frac{\|u''(t)\|}{\|u(t)\|} \\ &\leq \frac{2K_1}{\|u(t)\|^2} \|Au(t) - \frac{M(t)}{\|u(t)\|^2}u(t)\|^2 + \frac{\rho^2}{2K_1} \frac{\|u''(t)\|^2}{\|u(t)\|^2}, \end{aligned}$$

and moreover, by $a(M(t)) \geq K_1 > 0$,

$$\frac{d}{dt} \frac{M(t)}{\|u(t)\|^2} \leq C \frac{\|u''(t)\|^2}{\|u(t)\|^2} = C \frac{\|u''(t)\|}{M(t)} \frac{M(t)}{\|u(t)\|^2} \leq \nu \frac{M(t)}{\|u(t)\|^2}$$

with some $\nu \geq 0$, where we used (5.4). Therefore, we obtain

$$\frac{M(t)}{\|u(t)\|^2} \leq Ce^{\nu t}$$

and hence

$$\|u(t)\|^2 \geq Ce^{-\nu t} M(t) \geq Ce^{-\nu t} e^{-\alpha t} = Ce^{-\beta t}$$

with some $\beta \geq \alpha > 0$, where we used (5.1). \square

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